Vectors and Planes (3D Geometry):

Q) Two planes have equations $\prod_1 : 2x + y + z = 1$ and $\prod_2 : 3x + y - z = 2$ respectively.

(a) Find the equation of the line (L) of the intersection of the two planes.

(b) Prove that the equation of the plane $\prod_3$, which contains line (L) and is perpendicular to plane $\prod_1$, is $\prod_3 : x - 2z = 1$

(c) Points P$(-2, 4, 1)$ and Q lie in the plane $\prod_3$, with PQ perpendicular to $\prod_2$.

Determine point Q.

Solution:

Concept used:

1.) Equation of the line $\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \lambda \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$ or $r = a_x i + a_y j + a_z k + \lambda (b_x i + b_y j + b_z k)$

2.) Equation of the Plane $ax + by + cz + d = 0$ has direction ratios of the normal to the plane as $n < a \quad b \quad c >$

(a)

Plane $\prod_1 : 2x + y + z = 1$

Direction Ratios of the vector normal to plane is $n_1 < 2 \quad 1 \quad 1 >$

Plane $\prod_2 : 3x + y - z = 2$

Direction Ratios of the vector normal to plane is $n_2 < 3 \quad 1 \quad -1 >$

Consider a point on the line of intersection of the plane (1) and plane (2), with $z = 0$.

Solving $2x + y = 1 \quad \cdots (i)$

$3x + y = 2 \quad \cdots (ii)$

$x = 1 \quad and \quad y = -1$

Point on the line of intersection $(1, -1, 0)$

Line of intersection is perpendicular to the normal vectors of both the planes. The direction ratios of the line are given by

$$\bar{n} = \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 2i - 5j + k$$

Equation of line is $\begin{bmatrix}1\\-1\\0\end{bmatrix}+\lambda\begin{bmatrix}2\\-5\\1\end{bmatrix}$ or $r=i-j+\lambda(2i-5j+k)$

Or $\frac{x-1}{2}=\frac{y+1}{-5}=\frac{z-0}{1}=\lambda$

(b)

Normal vector of plane (1), $n_1 <2 \quad 1 \quad 1>$, and line are in plane (3)

$$\overline{n_3}=\begin{vmatrix}i & j & k\\2 & 1 & 1\\2 & -5 & 1\end{vmatrix}$$

$$=6i-12k$$

$$Or$$

$$\overline{n_3}=i-2k$$

Equation of plane (3) can be written as $\prod_3 : x-2z=d$

Plane passes through $(1,-1,0)$

$\Rightarrow 1-2\times 0=d$

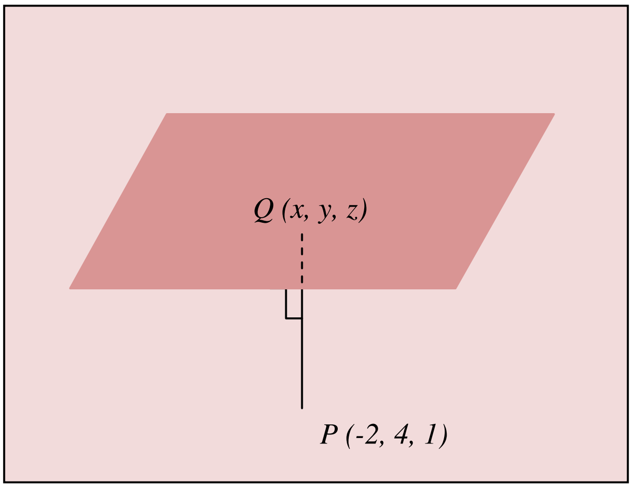
$\Rightarrow d=1$

Equation of plane (3) is $\prod_3 : x-2z=1$

(c)

Given $P(-2,4,1)$

Normal vector of plane (3), $n_3 <1 \quad 0 \quad -2>$



Line PQ is perpendicular to plane (3)

Therefore, Direction Ratios of line PQ are $n_3 < 1 \quad 0 \quad -2 >$

Equation of line is $\frac{x+2}{1} = \frac{y-4}{0} = \frac{z-1}{-2} = \mu$

Coordinates of Q can be written as

$\Rightarrow x = \mu - 2, \quad y = 0\mu + 4 = 4, \quad z = -2\mu + 1$

Point Q satisfies the plane (3) $\prod_3 : x - 2z = 1$

$$\Rightarrow (\mu - 2) - 2(-2\mu + 1) = 1$$
$$\Rightarrow 5\mu = 5$$
$$\Rightarrow \mu = 1$$

$$\Rightarrow x = 1 - 2 = -1 \quad y = 4, \quad z = -2(1) + 1 = -1$$
$$\Rightarrow Q \equiv (-1, 4, -1)$$