eMath Solutions

Sequences and Series along with Binomial Expansion (IIT)

Find the sum of the series $\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{(15)^r}{2^{4r}}+\cdots \textit{ up to Infinite Terms}\right)$

Solution:

Concepts used:

1.) Sum of infinite geometric series with first term as 'a' and common ratio 'r' is $S_\infty=\frac{a}{1-r}$

2.) Binomial Expansion in the sigma notation $(1-x)^n=\sum_{r=0}^{r=n}(-1)^r\,{}^nC_r x^r$

$$\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{(15)^r}{2^{4r}}+\cdots \textit{ up to Infinite Terms}\right)$$

$$=\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{4^r}+\frac{7^r}{8^r}+\frac{(15)^r}{(16)^r}+\cdots \textit{ up to Infinite Terms}\right]$$

$$=\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left[\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\cdots \textit{ up to Infinite Terms}\right]$$

$$=\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{3}{4}\right)^r+\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{7}{8}\right)^r+\sum_{r=0}^{r=n}(-1)^r\ {}^nC_r\left(\frac{15}{16}\right)^r+\cdots \textit{ up to Infinite Terms}$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\left(1-\frac{15}{16}\right)^n+\cdots \textit{ up to Infinite Terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \cdots \textit{ up to Infinite Terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n} + \left(\frac{1}{2}\right)^{3n} + \left(\frac{1}{2}\right)^{4n} + \cdots \textit{ up to Infinite Terms}$$

This represents an infinite geometric series with first term as $a = \frac{1}{2^n}$ and common ratio as $r = \frac{1}{2^n}$

$$S_\infty = \frac{\frac{1}{2^n}}{1 - \frac{1}{2^n}}$$

$$= \frac{1}{2^n - 1}$$