McLaren Series (Polynomial)

Q.) Compute the 6th degree McLaren polynomial for the function

$$f(x)=\log(\cos x), \qquad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

and hence evaluate $\int_0^{\frac{\pi}{4}} \log(\cos x)\,dx$

Solution:

Concepts used:

$(i)\quad \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\Lambda$

$(ii)\quad \cos x=1-\frac{x}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\Lambda$

$$f(x)=\log(\cos x), \quad , \quad , \quad , \quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\Rightarrow f(x)=\log[1+(\cos x-1)]$$

From the natural logarithm

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\Lambda \ (terms\ with\ powers\ of\ x\geq 4)$$

and for the cosine function

$$\cos x=1-\frac{x}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\Lambda$$

$$\Rightarrow \cos x-1=-\frac{x}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\Lambda \ (terms\ with\ powers\ of\ x\geq 8)$$

$$=-\frac{x}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\Lambda \ (terms\ with\ powers\ of\ x\geq 8)$$

The latter series expansion has a zero constant term, which enables us to substitute the second series into the first one and to easily omit terms of higher exponents than 7.

$$f(x) = \log[1 + (\cos x - 1)]$$

$$= (\cos x - 1) - \frac{1}{2}(\cos x - 1)^2 + \frac{1}{3}(\cos x - 1)^3 + \Lambda \ (\textit{terms with powers of } (\cos x - 1) \geq 4)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \Lambda \ (\textit{terms with powers of } x \geq 8) \right)$$

$$-\frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + \Lambda \ (\textit{terms with powers of } x \geq 6) \right)^2$$

$$+\frac{1}{3}\left(-\frac{x^2}{2} + \Lambda \ (\textit{terms with powers of } x \geq 4) \right)^3 + \Lambda \ (\textit{terms with powers of } x \geq 8)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^6}{24} + \Lambda \ (\textit{terms with powers of } x \geq 8)$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \Lambda \ (\textit{terms with powers of } x \geq 8)$$

Since the cosine is an even function, the coefficients for all the odd powers *x*, x^3, x^5, x^7, have value zero.

$$\int_0^{\frac{\pi}{4}} \log(\cos x)\,dx = \int_0^{\frac{\pi}{4}} \left(-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} \right) dx$$

$$= \left[-\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{315} \right]_0^{\frac{\pi}{4}}$$

$$= \left[\left(-\frac{x^3}{3} \right) \left(\frac{1}{2} + \frac{x^2}{20} + \frac{x^4}{105} \right) \right]_0^{\frac{\pi}{4}}$$

$$= (-0.1615)(0.5 + 0.0308 + 0.0036)$$

$$= -0.0863$$