Mathematical Induction (IB – HL)

Q.) Prove by method of Induction, for any positive integer

$$\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda \times \cos\left(2^{n-1}x\right) = \frac{\sin\left(2^n x\right)}{2^n \sin x}$$

Solution:

Concept used: Principle of Mathematical Induction:

For a preposition $P(n)$ to be true $\forall\ n \in Z^+$, the principle of mathematical induction states that "If $P(n)$ is true for $n = 1$ and assuming the truth value of $P(k),\ k \in Z^+$, if it can be proved that $P(k+1)$ is true, then $P(n)$ is true $\forall\ n \in Z^+$."

$$Let \quad P(n) \equiv \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda \times \cos\left(2^{n-1}x\right) = \frac{\sin\left(2^n x\right)}{2^n \sin x}, \quad n \in Z^+$$

Stage I : We prove the truth of $P(1)$

$$LHS\ of\ P(1) = \cos x$$

$$RHS\ of\ P(1) = \frac{\sin\left(2^1 x\right)}{2^1 \sin x} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x \cos x}{2\sin x} = \cos x$$

$$\Rightarrow LHS\ of\ P(1) = RHS\ of\ P(1)$$

$$\Rightarrow P(1)\ is\ true.$$

Stage II : We assume the truth of $P(k),\quad k \in Z^+$

$$P(k) \equiv \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda \times \cos\left(2^{k-1}x\right) = \frac{\sin\left(2^k x\right)}{2^k \sin x}, \quad k \in Z^+$$

Stage III : With the truth of $P(k)$, *we prove the truth of* $P(k+1)$

$$P(k+1) \equiv \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda \times \cos\left(2^{(k+1)-1}x\right) = \frac{\sin\left(2^{k+1} x\right)}{2^{k+1} \sin x}, \quad k \in Z^+$$

$$\Rightarrow P(k+1) \equiv \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda \times \cos\left(2^{k}x\right) = \frac{\sin\left(2^{k+1} x\right)}{2^{k+1} \sin x}, \quad k \in Z^+$$

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\begin{aligned}
LHS\ of\ \ P(k+1) &= \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda\ \times \cos\left(2^{k} x\right) \\
&= \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda\ \times \cos\left(2^{k-1} x\right) \times \cos\left(2^{k} x\right) \\
&= \left[\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \Lambda\ \times \cos\left(2^{k-1} x\right)\right] \times \cos\left(2^{k} x\right) \\
&= \frac{\sin\left(2^{k} x\right)}{2^{k} \sin x} \times \cos\left(2^{k} x\right) \\
&= \frac{2 \sin\left(2^{k} x\right) \cos\left(2^{k} x\right)}{2^{k+1} \sin x} \\
&= \frac{\sin\left(2^{k+1} x\right)}{2^{k+1} \sin x} \\
&= RHS\ of\ \ P(k+1)
\end{aligned}
$$

$\therefore\ \ Truth\ Value\ of\ \ P(k) \Rightarrow Truth\ Value\ of\ \ P(k+1)$

$\therefore\ \ Truth\ Value\ of\ \ P(1) \Rightarrow Truth\ Value\ of\ \ P(2)$

$Similarly\ \ \ Truth\ Value\ of\ \ P(2) \Rightarrow Truth\ Value\ of\ \ P(3)$

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and so on

Hence, by the Principle of Mathematical Induction,

$P(n)\ is\ true\ \forall\ n \in Z^{+}$