eMath Solutions

Complex Exponent of a Complex Number (Engineering Math)

To evaluate: $(a+ib)^{(c+id)}$

Solution:

Let $z = a+ib$

$$= re^{i\theta} \qquad where\ r=\sqrt{a^2+b^2}\ and\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow \log z = \log(a+ib)$$
$$= (\log r) + i\theta$$
$$= \frac{1}{2}\log(a^2+b^2) + i\theta$$

$$\Rightarrow z = e^{\frac{1}{2}\log(a^2+b^2)+i\theta}$$
$$= e^{\log\sqrt{a^2+b^2}+i\theta}$$
$$= \sqrt{a^2+b^2}\times e^{i\theta}$$

$$z = (a+ib)^{(c+id)}$$
$$= \left[\sqrt{a^2+b^2}\times e^{i\theta}\right]^{(c+id)}$$
$$= \left(\sqrt{a^2+b^2}\right)^{c+id}\times\left(e^{i\theta}\right)^{c+id}$$
$$= \left((a^2+b^2)^{\frac{c}{2}}\times(a^2+b^2)^{\frac{1}{2}id}\right)\times e^{i\theta c}\times e^{-\theta d}$$
$$= (a^2+b^2)^{\frac{c}{2}}e^{-\theta d}\left[(a^2+b^2)^{\frac{1}{2}id}\times e^{i\theta c}\right]$$
$$= (a^2+b^2)^{\frac{c}{2}}e^{-\theta d}\left[r^{id}\times e^{i\theta c}\right]$$
$$= (a^2+b^2)^{\frac{c}{2}}e^{-\theta d}\left[e^{id\log r}\times e^{i\theta c}\right]$$
$$= (a^2+b^2)^{\frac{c}{2}}e^{-\theta d}\left[e^{i(c\theta+d\log r)}\right]$$
$$= (a^2+b^2)^{\frac{c}{2}}e^{-\theta d}\left[\cos(c\theta+d\log r)+i\sin(c\theta+d\log r)\right]$$

Example: $\left(\sqrt{i}\right)^{\sqrt{i}}$

$$\left(\sqrt{i}\right)^{\sqrt{i}} \quad\Rightarrow\quad a+ib=\sqrt{i} \quad and \quad c+id=\sqrt{i}$$

$for\ a+ib=\sqrt{i},\quad r=a^2+b^2=1\ and\ \theta=\frac{\pi}{4},\frac{5\pi}{4}$ (Applying D'Moivre's Theorem)

$$\Rightarrow a+ib=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}} \quad or \quad a+ib=-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$$

Similarly, $c+id=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}} \quad or \quad c+id=-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$

We have four cases

(1) $\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$ (2) $\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$

(3) $\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}$ (4) $\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}$

We also have $r=1 \quad \Rightarrow \quad a^2+b^2=1$

And $\log r=\log 1=0$

Case (1): $\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$

$$z=\left(a^2+b^2\right)^{\frac{c}{2}}e^{-\theta d}\left[\cos(c\theta+d\log r)+i\sin(c\theta+d\log r)\right]$$

$$\theta=\frac{\pi}{4} \quad and \quad c=\frac{1}{\sqrt{2}},\ d=\frac{1}{\sqrt{2}}$$

$$\Rightarrow z_1=1\times e^{-\left(\frac{\pi}{4\sqrt{2}}\right)}\left[\cos\left\{\left(\frac{1}{\sqrt{2}}\times\frac{\pi}{4}\right)+0\right\}+i\sin\left\{\left(\frac{1}{\sqrt{2}}\times\frac{\pi}{4}\right)+0\right\}\right]$$

$$=e^{-\left(\frac{\pi}{4\sqrt{2}}\right)}\left[\cos\left(\frac{\pi}{4\sqrt{2}}\right)+i\sin\left(\frac{\pi}{4\sqrt{2}}\right)\right]$$

Case (2): $\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}$

$$z=\left(a^2+b^2\right)^{\frac{c}{2}}e^{-\theta d}\left[\cos(c\theta+d\log r)+i\sin(c\theta+d\log r)\right]$$

$$\theta=\frac{5\pi}{4} \quad and \quad c=\frac{1}{\sqrt{2}},\ d=\frac{1}{\sqrt{2}}$$

$$\Rightarrow z_2=1\times e^{-\left(\frac{5\pi}{4\sqrt{2}}\right)}\left[\cos\left\{\left(\frac{1}{\sqrt{2}}\times\frac{5\pi}{4}\right)+0\right\}+i\sin\left\{\left(\frac{1}{\sqrt{2}}\times\frac{5\pi}{4}\right)+0\right\}\right]$$

$$=e^{-\left(\frac{5\pi}{4\sqrt{2}}\right)}\left[\cos\left(\frac{5\pi}{4\sqrt{2}}\right)+i\sin\left(\frac{5\pi}{4\sqrt{2}}\right)\right]$$

Case (3): $\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}$

$$z=\left(a^2+b^2\right)^{\frac{c}{2}}e^{-\theta d}\left[\cos\left(c\theta+d\log r\right)+i\sin\left(c\theta+d\log r\right)\right]$$

$$\theta=\frac{\pi}{4}\quad and\quad c=-\frac{1}{\sqrt{2}},\, d=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow z_3=1\times e^{\left(\frac{\pi}{4\sqrt{2}}\right)}\left[\cos\left\{\left(-\frac{1}{\sqrt{2}}\times\frac{\pi}{4}\right)+0\right\}+i\sin\left\{\left(-\frac{1}{\sqrt{2}}\times\frac{\pi}{4}\right)+0\right\}\right]$$

$$=e^{\left(\frac{\pi}{4\sqrt{2}}\right)}\left[\cos\left(\frac{\pi}{4\sqrt{2}}\right)-i\sin\left(\frac{\pi}{4\sqrt{2}}\right)\right]$$

Case (4): $\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)}$

$$z=\left(a^2+b^2\right)^{\frac{c}{2}}e^{-\theta d}\left[\cos\left(c\theta+d\log r\right)+i\sin\left(c\theta+d\log r\right)\right]$$

$$\theta=\frac{5\pi}{4}\quad and\quad c=-\frac{1}{\sqrt{2}},\, d=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow z_4=1\times e^{\left(\frac{5\pi}{4\sqrt{2}}\right)}\left[\cos\left\{\left(-\frac{1}{\sqrt{2}}\times\frac{5\pi}{4}\right)+0\right\}+i\sin\left\{\left(-\frac{1}{\sqrt{2}}\times\frac{5\pi}{4}\right)+0\right\}\right]$$

$$=e^{\left(\frac{5\pi}{4\sqrt{2}}\right)}\left[\cos\left(\frac{5\pi}{4\sqrt{2}}\right)-i\sin\left(\frac{5\pi}{4\sqrt{2}}\right)\right]$$