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Calculus

Applications of Derivatives

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Applications of Derivatives

The concept of Applications of Derivatives is studied under the following sub – concepts:

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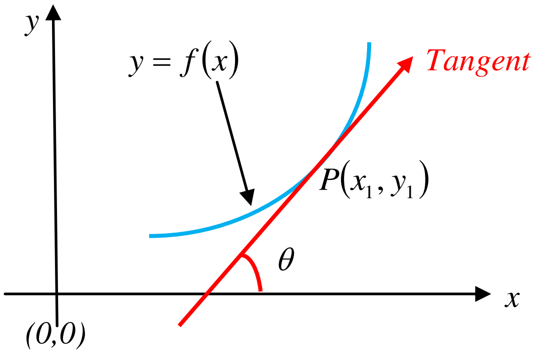
(1) <u>*Tangents and Normals:*</u>

Let $y = f(x)$ represents the equation of the curve

Then by the definition of the derivative,

$$\left.\frac{dy}{dx}\right|_{P(x_1,y_1)} = f'(x)\Big|_{P(x_1,y_1)}$$

Represent the slope of the tangent drawn to the curve at point $P(x_1, y_1)$



Equation of the tangent in Slope – Point form is

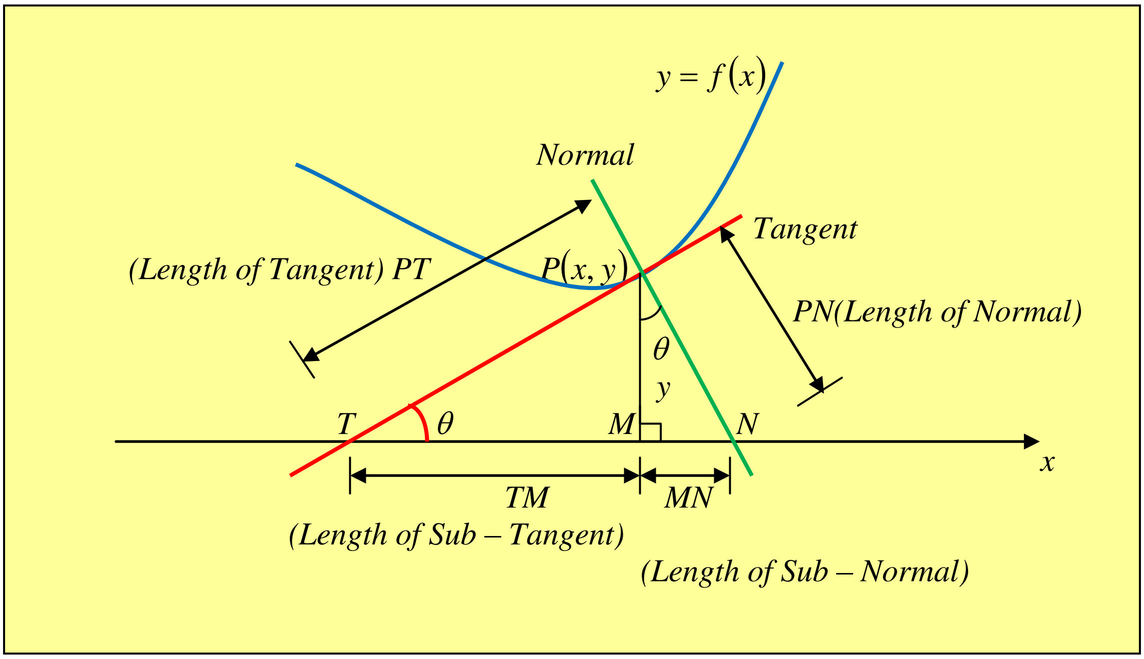
$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} (x - x_1)$$

Equation of the normal in Slope – Point form is

$$y - y_1 = \frac{-1}{\left.\frac{dy}{dx}\right|_{(x_1,y_1)}} (x - x_1)$$

(2) <u>*Length of Tangent, Normal, Sub – tangent and Sub – normal:*</u>

Consider function $y = f(x)$



We have, from the diagram, for point $P(x, y)$

$$PM = y$$

$PT = Length\ of\ the\ Tangent$
$PN = Length\ of\ the\ Normal$
$TM = Length\ of\ the\ Sub-Tangent$
$MN = Length\ of\ the\ Sub\text{ - }Normal$

Slope of the tangent at $P(x, y)$, *given by*

$$m = \tan\theta = \left.\frac{dy}{dx}\right|_P$$

$$\frac{PM}{TM} = \tan\theta = \left.\frac{dy}{dx}\right|_P$$

$$\Rightarrow \frac{y}{TM} = \left.\frac{dy}{dx}\right|_P$$

$$\Rightarrow TM = \frac{y}{\left(\left.\frac{dy}{dx}\right|_P\right)}$$

$$Length\ of\ the\ Sub-Tangent = \frac{y}{\left(\left.\frac{dy}{dx}\right|_P\right)}$$

From right – angled triangle ΔPMT

$$PT^2 = PM^2 + TM^2$$

$$= y^2 + \frac{y^2}{\left(\left.\frac{dy}{dx}\right|_P\right)^2}$$

$$PT = y \times \sqrt{1 + \frac{1}{\left(\left.\frac{dy}{dx}\right|_P\right)^2}}$$

$$Length\ of\ the\ Tangent = y \times \sqrt{1 + \frac{1}{\left(\left.\frac{dy}{dx}\right|_P\right)^2}}$$

In right – angled triangle ΔPMN, $\angle MPN = \theta$

$$\frac{MN}{PM} = \tan\theta = \left.\frac{dy}{dx}\right|_P$$

$$\Rightarrow \frac{MN}{y} = \left.\frac{dy}{dx}\right|_P$$

$$\Rightarrow MN = y \times \left(\left.\frac{dy}{dx}\right|_P\right)$$

$$Length\ of\ the\ Sub\text{ - }Normal = y \times \left(\left.\frac{dy}{dx}\right|_P\right)$$

Also,

$$PN^2 = PM^2 + MN^2$$

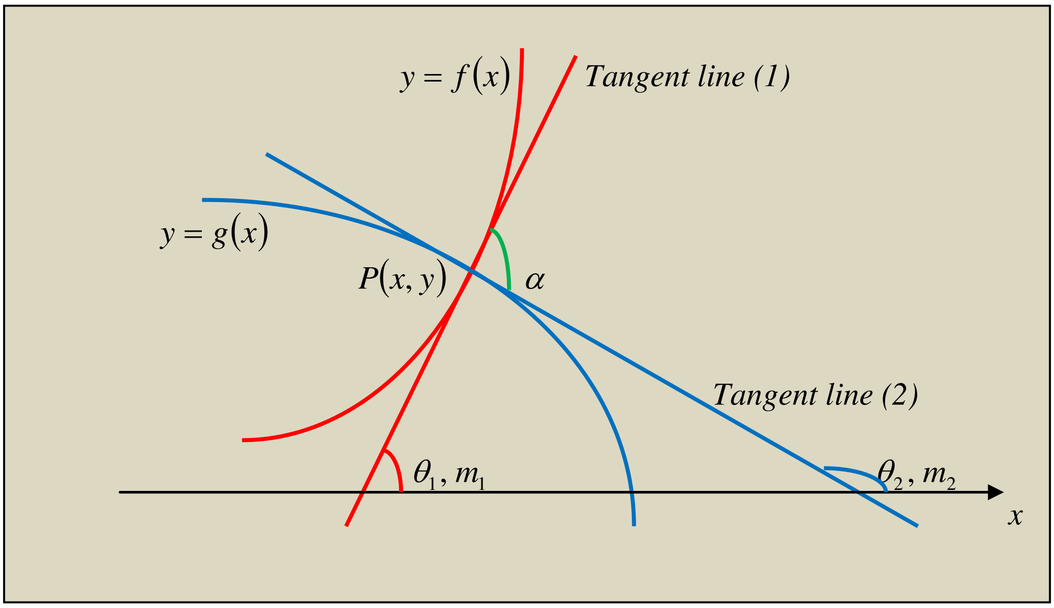
$$= y^2 + y^2\left(\left.\frac{dy}{dx}\right|_P\right)^2$$

$$PN = y \times \sqrt{1 + \left(\left.\frac{dy}{dx}\right|_P\right)^2}$$

$$Length\ of\ the\ Normal = y \times \sqrt{1 + \left(\left.\frac{dy}{dx}\right|_P\right)^2}$$

(3) Angle of Intersection of the curves:

Angle of intersection of two curves, represented by functions $y = f(x)$ and $y = g(x)$ is defined as the angle between the tangents drawn to tow curves at the point of intersection.



If the tangent line drawn to the curve $y = f(x)$ at point $P(x, y)$ makes an angle θ_1 with the positive direction of x – axis, then slope m_1 of the tangent line is given by

$$m_1 = \tan\theta_1 = \left.\frac{df}{dx}\right|_P$$

Similarly, If the tangent line drawn to the curve $y = g(x)$ at point $P(x, y)$ makes an angle θ_2 with the positive direction of x – axis, then slope m_2 of the tangent line is given by

$$m_2 = \tan\theta_2 = \left.\frac{dg}{dx}\right|_P$$

If α is the angle between the two tangents drawn at point of intersection $P(x, y)$, then

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\left.\frac{df}{dx}\right|_P - \left.\frac{dg}{dx}\right|_P}{1 + \left.\frac{df}{dx}\right|_P \left.\frac{dg}{dx}\right|_P}\right|$$

Orthogonal Intersection:

Curves are said to intersect orthogonality or intersection is said to be orthogonal, if the angle of intersection is 90^0 or $\frac{\pi}{2}$.

That is $\qquad \alpha = \frac{\pi}{2}$

That is $\qquad 1 + \left.\frac{df}{dx}\right|_P \left.\frac{dg}{dx}\right|_P = 0$

$$\Rightarrow \left.\frac{df}{dx}\right|_P \left.\frac{dg}{dx}\right|_P = -1$$

Important Result:

If $\qquad S_1 : x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$

And $\qquad S_2 : x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$

Represent equations of two circles which intersect orthogonally, then

$$2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$$

(4) Increasing Decreasing Functions:

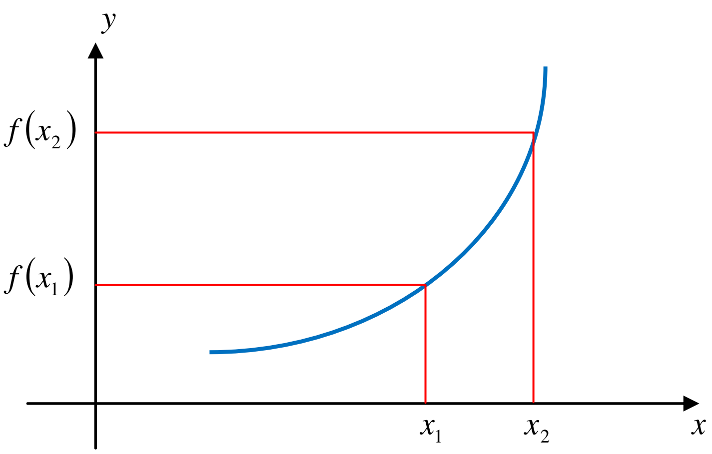
Increasing Function

$$\begin{aligned}
&\textit{Given function } y = f(x)\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1) \quad \textit{then } f(x) \textit{ is a non-decreasing function.}\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) > f(x_1) \quad \textit{then } f(x) \textit{ is an increasing function.}
\end{aligned}$$

OR

$$\begin{aligned}
&\textit{Given function } y = f(x)\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1) \quad \textit{then } f(x) \textit{ is an increasing function.}\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) > f(x_1) \quad \textit{then } f(x) \textit{ is strictly (monotonically)}\\
&\qquad \qquad \qquad \qquad \qquad \qquad \qquad \textit{increasing function.}
\end{aligned}$$

Graph of an increasing function:



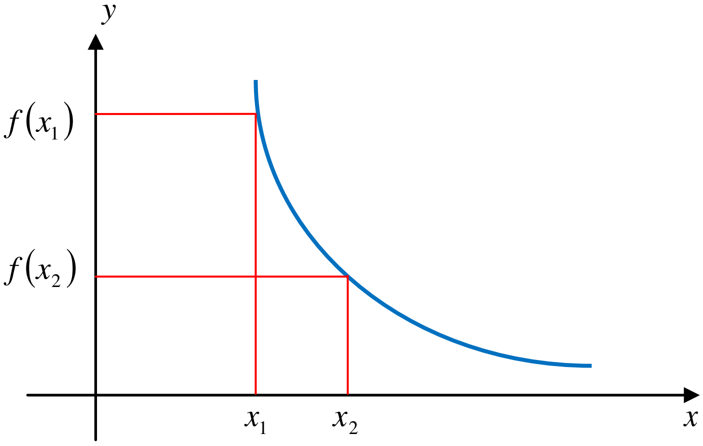
Decreasing Function

$$\begin{aligned}
&\textit{Given function } y = f(x)\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1) \quad \textit{then } f(x) \textit{ is a non-increasing function.}\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) < f(x_1) \quad \textit{then } f(x) \textit{ is a decreasing function.}
\end{aligned}$$

OR

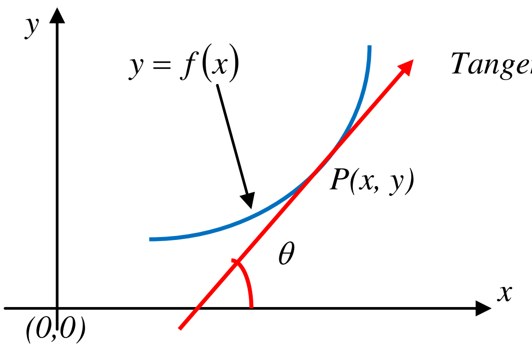
$$\begin{aligned}
&\textit{Given function } y = f(x)\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1) \quad \textit{then } f(x) \textit{ is a decreasing function.}\\
&\textit{If } \quad x_2 > x_1 \Rightarrow f(x_2) > f(x_1) \quad \textit{then } f(x) \textit{ is strictly (monotonically)}\\
&\qquad \qquad \qquad \qquad \qquad \qquad \qquad \textit{decreasing function.}
\end{aligned}$$

Graph of an decreasing function:



<u>Relation between</u> $\tan\theta$ *and* $\frac{dy}{dx} / f'(x)$

- Angle of inclination θ of tangent line drawn to graph of the function $y = f(x)$ at point $P(x, y)$.
- Correlation of $\tan\theta$ *to* $\frac{dy}{dx} / f'(x)$

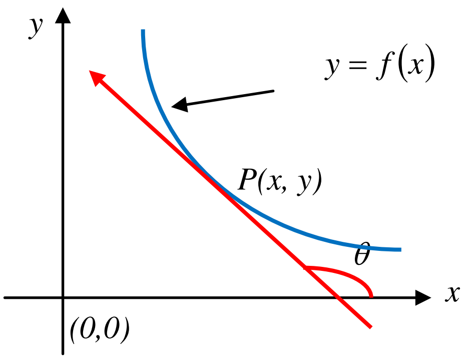


Increasing function

θ *is acute*

$$\tan\theta = \left.\frac{dy}{dx}\right|_{P(x,y)} > 0$$

(Slope of the tangent line is positive)



Decreasing function

θ *is obtuse*

$$\tan\theta = \left.\frac{dy}{dx}\right|_{P(x,y)} < 0$$

(Slope of the tangent line is negative)

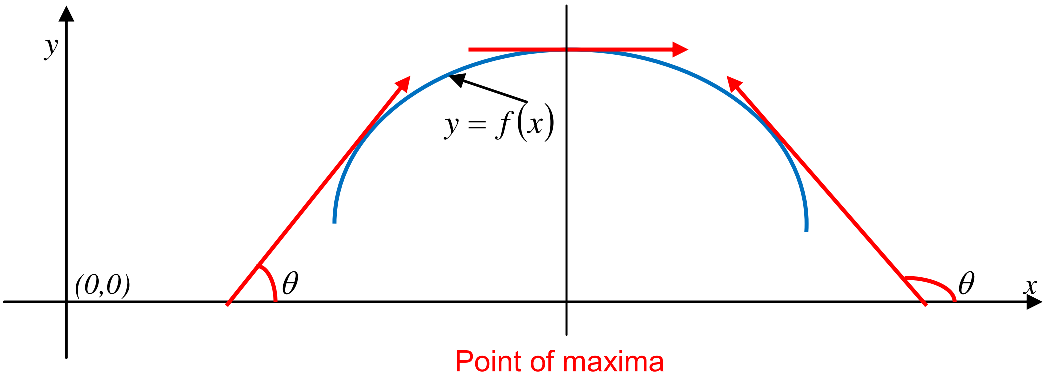
(5) Stationery Points:

(i) Maxima – Minima (Global Maxima / Global Minima and Local Maxima / Local Minima or Absolute Maxima / Absolute Minima and Relative Maxima / Relative Minima)

(ii) Point of Inflexion, Concavity / Convexity of the Curves

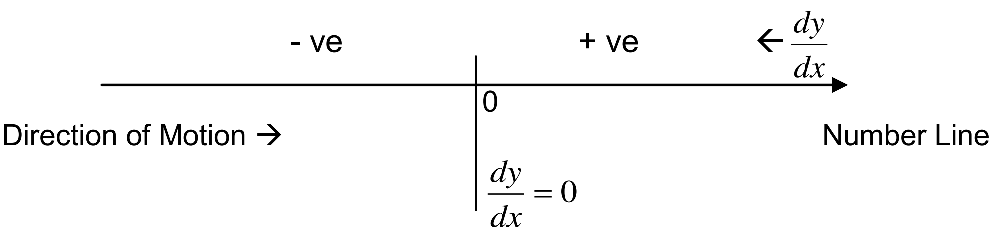
1.) Point of Maxima → $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$

$\frac{dy}{dx} = 0 \Rightarrow$ Tangent parallel to x – axis



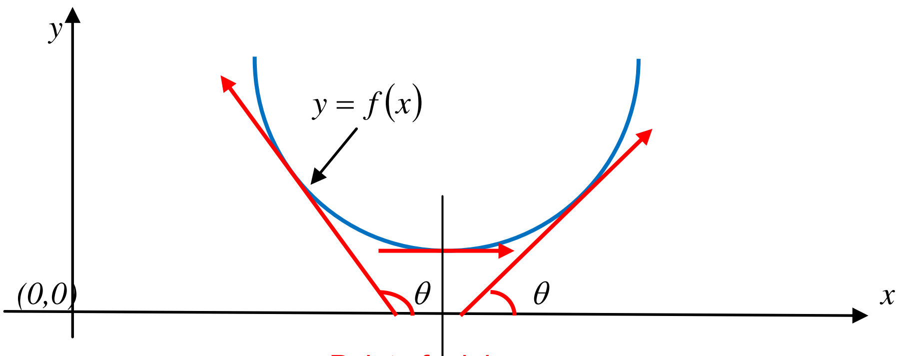
Point of maxima

Maxima on the Number Line:



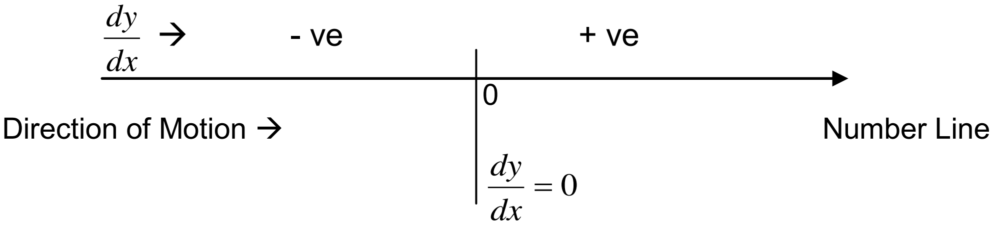
2.) Point of Minima → $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$

$\frac{dy}{dx} = 0 \Rightarrow$ Tangent parallel to x – axis

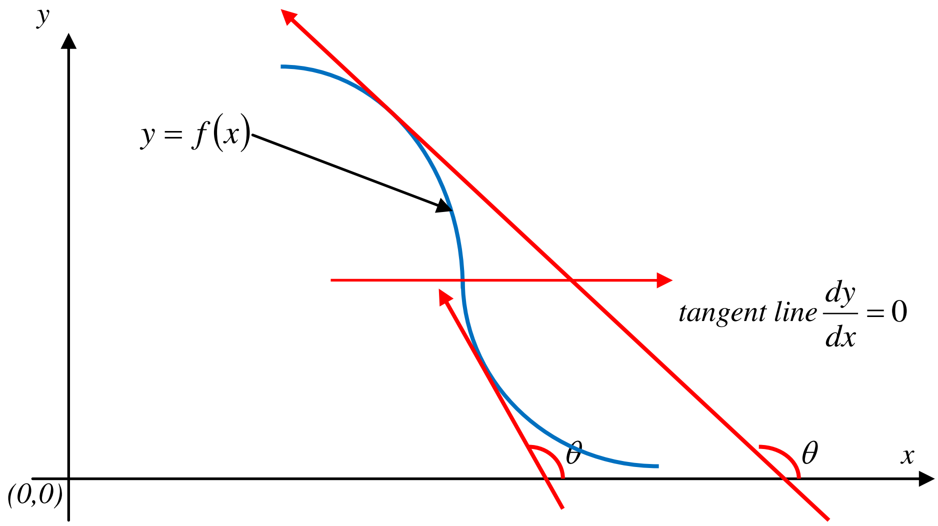


Point of minima

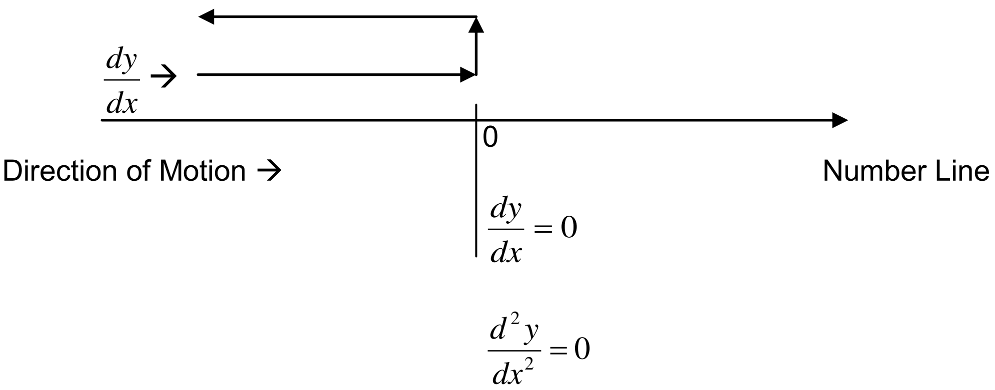
Minima on the Number Line:



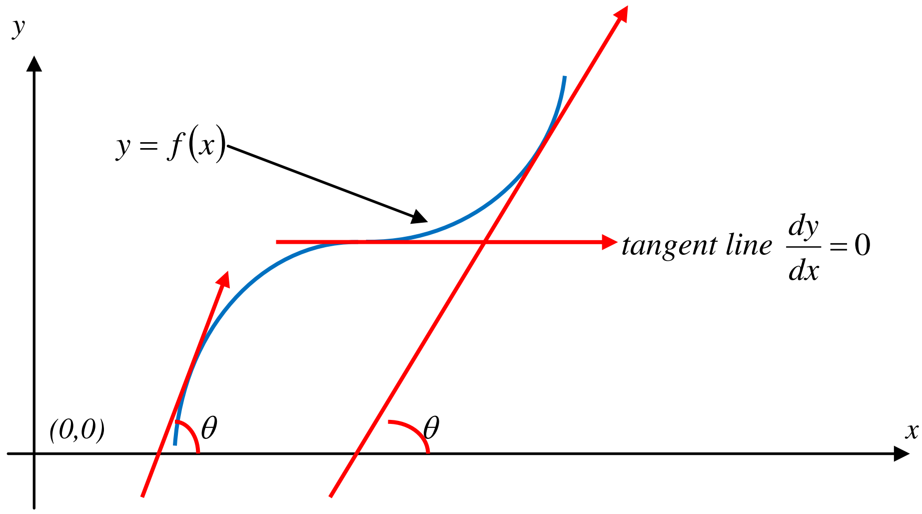
3.) Point of Inflexion → $\frac{dy}{dx}=0,\ \frac{d^2y}{dx^2}=0$



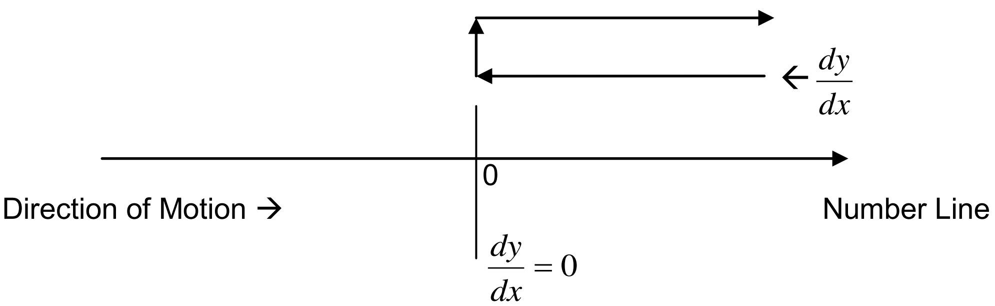
Point of Inflexion on Number Line:
(For the diagram as shown above



Point of Inflexion → $\frac{dy}{dx}=0, \frac{d^2y}{dx^2}=0$



Point of Inflexion on Number Line:
(For the diagram as shown above



(6) Shortest distance

(a) Between a curve and a point
(b) Between two curves (Non intersecting)

Important: Shortest distance is the normal distance from the given point to the curve or between the two curves.

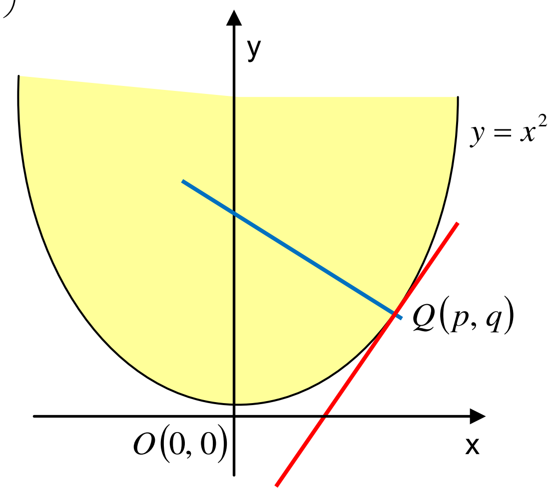
(a) Shortest distance between a curve and a point:

Example (): The shortest distance between the curve $y = x^2$ and $P\left(0, \frac{5}{2}\right)$ is ________

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) $\frac{3}{2}$

Solution: option (d)

Curve $y = x^2$ and $P\left(0, \frac{5}{2}\right)$



Let $Q(p, q)$ be a point on the curve $y = x^2$, such that PQ is the shortest distance.

PQ is the shortest distance $\Rightarrow$ PQ is normal to parabola $y = x^2$.

$$y = x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x$$

Also $Q(p, q)$ lies on $y = x^2 \Rightarrow q = p^2$

$\Rightarrow$ Slope of tangent at $Q(p, q)$ is $m_1 = 2p$

$\Rightarrow$ Slope of normal at $Q(p, q)$ is $m_2 = -\frac{1}{2p}$

Equation of normal at $Q(p, q)$ is $\quad y - p^2 = -\frac{1}{2p}(x - p)$

This passes through $P\left(0, \frac{5}{2}\right) \Rightarrow \frac{5}{2} - p^2 = -\frac{1}{2p}(0 - p)$

$\Rightarrow 5 - 2p^2 = 1$

$\Rightarrow 2p^2 = 4$

$\Rightarrow p = \pm\sqrt{2}$

$p = \pm\sqrt{2} \;\Rightarrow\; q = p^2 = 2$

$\Rightarrow$ There are two points $Q\left(\sqrt{2}, 2\right)$ in first quadrant and $Q'\left(-\sqrt{2}, 2\right)$ in the second quadrant.

Distance $PQ = \sqrt{\left(\sqrt{2}-0\right)^2 + \left(2-\frac{5}{2}\right)^2}$

$= \sqrt{2+\frac{1}{4}}$

$= \sqrt{\frac{9}{4}}$

$= \frac{3}{2}\,units$

Example (): The shortest distance between the curve $y^2 = 4x$ and $x^2 + (y-3)^2 = 4$ is ________

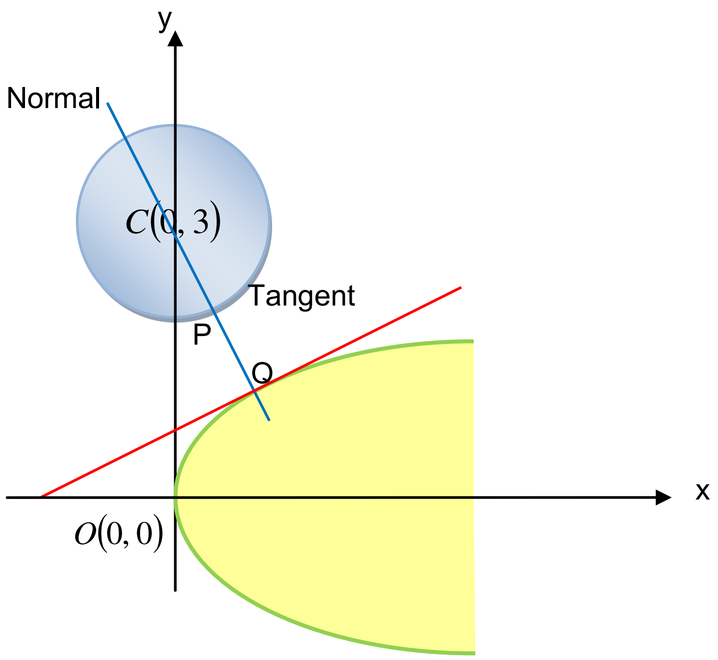
(a) $2-\sqrt{2}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

Solution: option (a)

$y^2 = 4x \Rightarrow$ Parabola with vertex $O(0, 0)$

$x^2 + (y-3)^2 = 4 \Rightarrow$ Circle with Centre $C(0, 3)$ and radius $r = 2$

Shortest distance is along the line which is normal to both the curves.



From the diagram, CPQ is the normal to both the curves. Also, normal passes through the centre of the circle. PQ is the shortest distance.
Let $Q(p, 2p)$ on the parabola.

$$y^2 = 4x \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4$$

$$Q(2, 2p) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{2p} = \frac{1}{p}$$

Slope of tangent at Q is $m = \frac{1}{p}$

Therefore, slope of normal at Q is $m' = -p$

. Equation of normal at Q is

$$y - 2p = -p(x - p)$$

This passes through $C(0, 3)$

$\Rightarrow 3 - 2p = -p(0 - p)$

$\Rightarrow p^2 + 2p - 3 = 0$

$\Rightarrow (p+3)(p-1) = 0$

$\Rightarrow p = -3, 1$

$p = -3$ is rejected (p cannot be negative)

$\Rightarrow p = 1$

$\Rightarrow Q(1, 2)$

$CQ = \sqrt{(1-0)^2 + (3-2)^2} = \sqrt{2}$

$$\begin{aligned} PQ &= |\, CQ - CP \,| \\ &= |\, \sqrt{2} - 2 \,| \\ &= 2 - \sqrt{2} \end{aligned}$$

(7) Mean Value Theorems:

(a) Lagrange's Mean Value Theorem (LMVT):

Statement: *Let function* $f(x)$ *be*

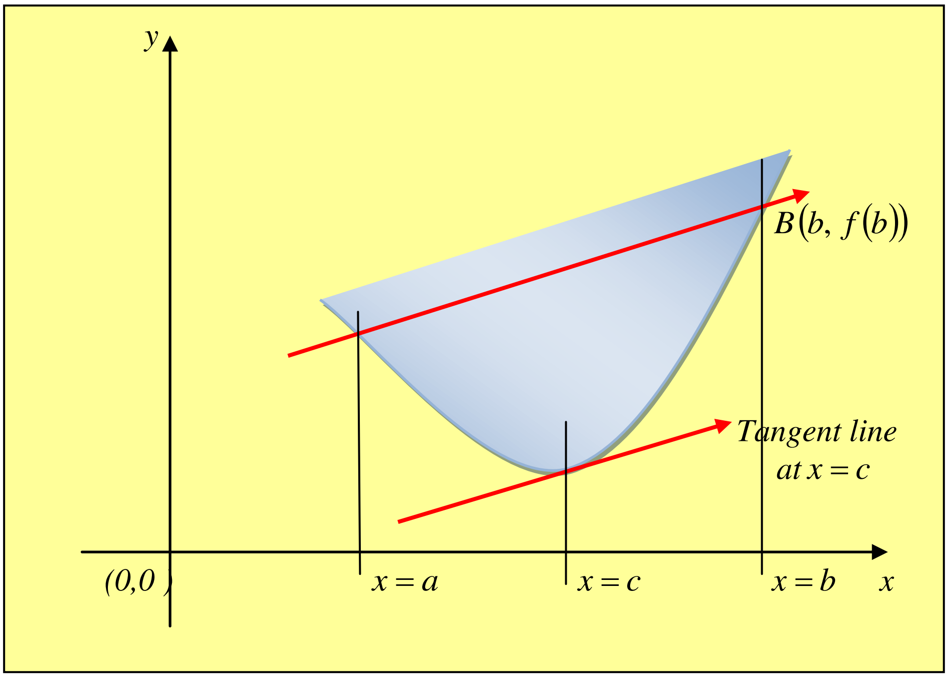
(i) Continuous in $[a,b]$

(ii) Differentiable in (a,b)

Then, $\exists$ *atleast one point* $x=c, c\in(a,b)$

Such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Geometrical Meaning of LMVT:



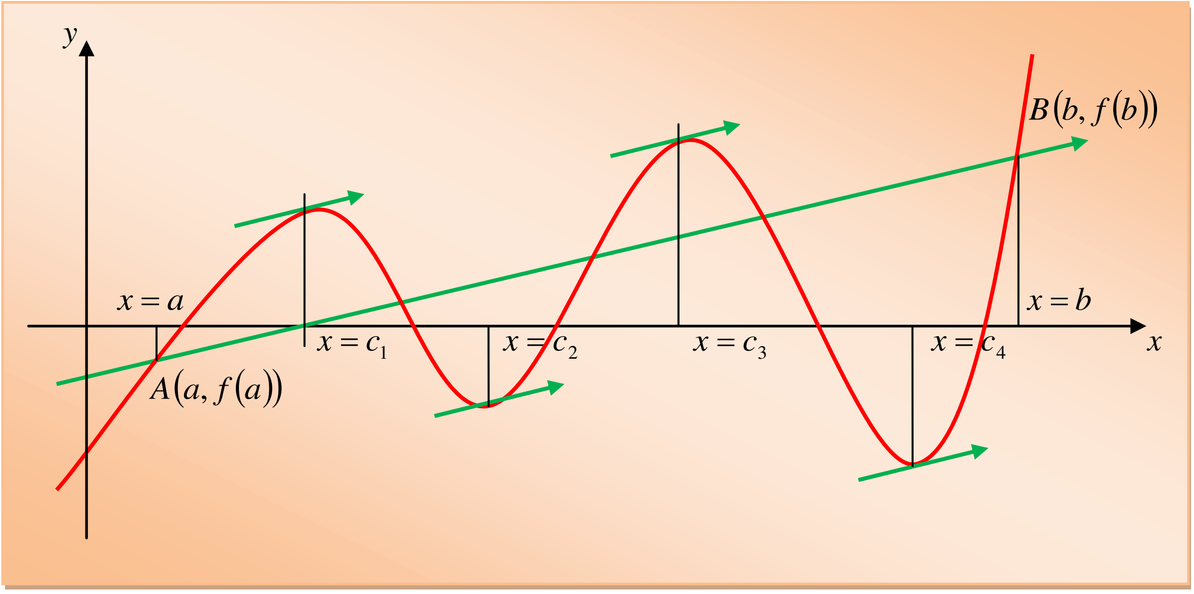
According to Lagrange's Mean Theorem,

Slope of the tangent line (at $x=c$) = slope of the chord AB

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

That is, there exists a point $x=c, c\in(a,b)$, such that tangent drawn at $x=c$ is parallel to the chord AB.

Very Important:
There can be more than one point where tangents drawn will be parallel to chord AB



From the diagram,

$$f'(c_1) = f'(c_2) = f'(c_3) = f'(c_4) = \frac{f(b) - f(a)}{b - a}$$

(b) Rolle's Theorem:

Statement: *Let function* $f(x)$ *be*

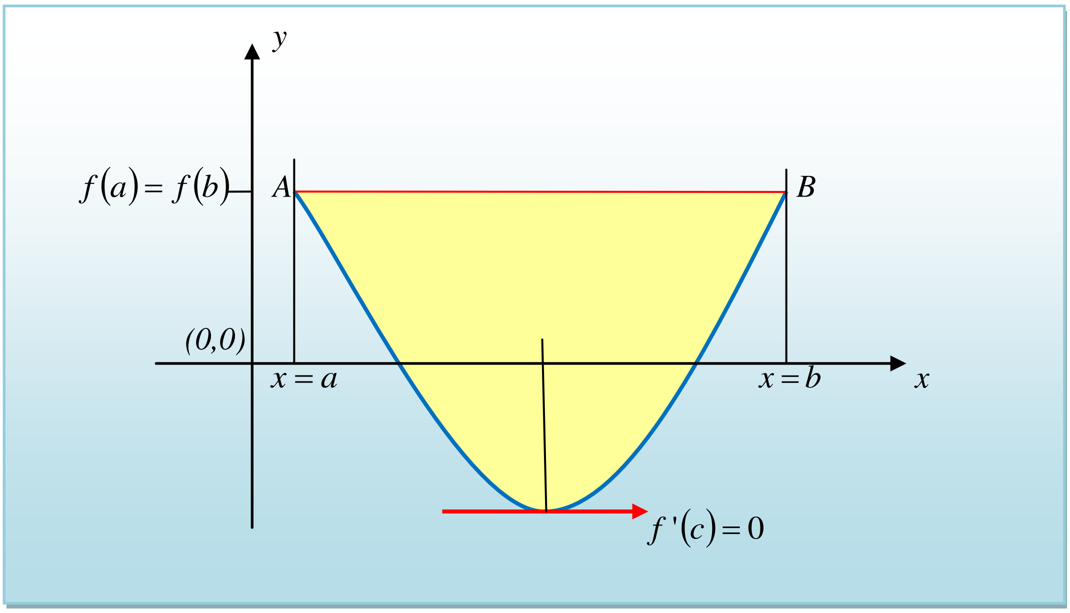
(i) Continuous in $[a, b]$

(ii) Differentiable in (a, b)

(iii) $f(b) = f(a)$

Then, $\exists$ *atleast one point* $x = c, c \in (a, b)$

Such that $f'(c) = 0$



Important:

- Geometrical Meaning of Rolle's Theorem:

  There exists a point $x = c, c \in (a, b)$, such that tangent drawn at $x = c$ is parallel to x - axis.

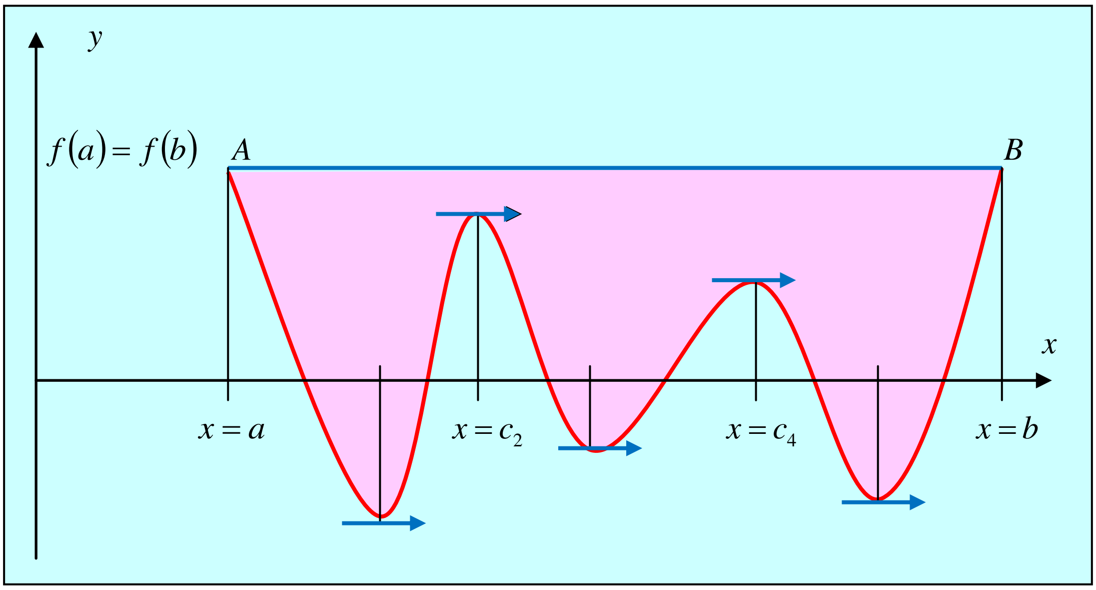
- Rolle's Theorem can be derived from Lagrange's Mean Value Theorem.
  According to Lagrange's Mean Theorem,
  Slope of the tangent line (at $x = c$) = slope of the chord AB

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(b) = f(a) \quad \Rightarrow \quad f'(c) = 0$$

  That is tangent at $x = c$ is parallel to x – axis.

- There may exist more than one point, where $f'(c) = 0$



From the diagram,

$$f'(c_1) = f'(c_2) = f'(c_3) = f'(c_4) == f'(c_5) = 0$$

Relation between the Roots of an equation and Rolle's Theorem:

Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a \neq 0$

Let $f(x) = 0$ has two real roots α *and* β.

Then from Rolle's Theorem,

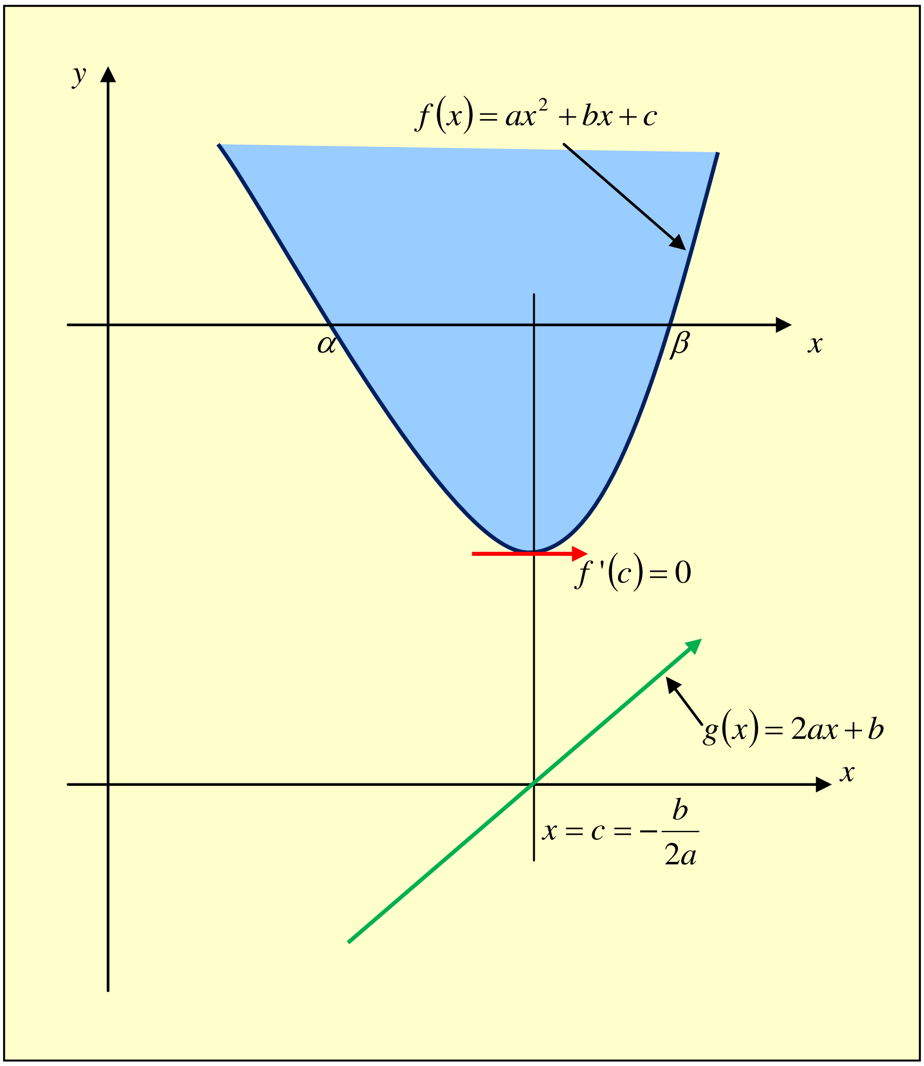
$f'(x) = 0$ will have a root between α *and* β

$f'(x) = 0$

$\Rightarrow 2ax + b = 0$

$\Rightarrow c = -\dfrac{b}{2a}, \quad c \in (a,b)$

Where, $x = c$ is root of the equation $2ax + b = 0$



$x = c = -\dfrac{b}{2a}$ is the root of $g(x) = 0$

Where, $g(x) = f'(x) = 2ax + b$

This conjecture holds true for a polynomial function of degree $"n"$

$$f(x) = p(x)$$

having $"n"$ real roots.

(c) Cauchy's Mean Value Theorem:

Statement: $Let\ functions\ f(x)\ and\ g(x)\ be$

(i) Continuous in $[a,b]$

(ii) Differentiable in (a,b)

(iii) $g'(x) \neq 0 \quad \forall\ x \in R$

And (iv) $g(b) \neq g(a)$

Then, $\exists$ $atleast\ one\ point\ x = c,\ c \in (a,b)$

Such that, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

(8) Derivatives and the roots of a polynomial equation

Application of derivatives helps in:

- Evaluation of roots
- Location of roots

Evaluation of Roots:

Example (): Two roots of the cubic equation $ax^3 + 3bx^2 + 3cx + d = 0$ are equal. Then each of the equal roots is ___________

(a) $\dfrac{ac - bd}{2(ac - b^2)}$ (b) $\dfrac{bc - ad}{2(bc - d^2)}$

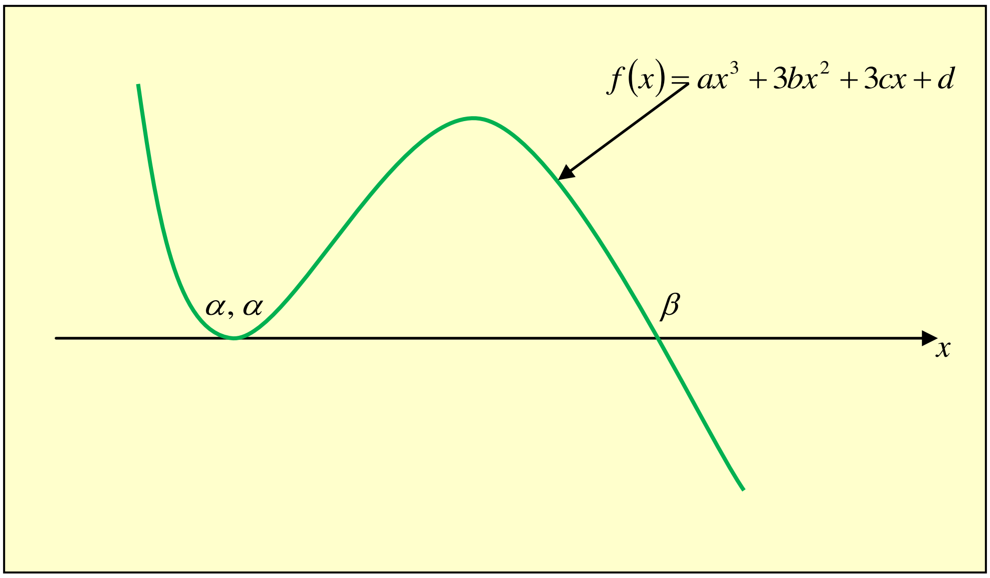
(c) $\dfrac{bc - ad}{2(ac - b^2)}$ (d) $\dfrac{ac - bd}{2(b^2 - ac)}$

Solution: option (c)

$ax^3 + 3bx^2 + 3cx + d = 0$

Let the roots of the equation be $\alpha,\ \alpha\ and\ \beta$.

$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \qquad \cdots(i)$



α is a repeated root

$\Rightarrow f'(\alpha) = 0$

$\Rightarrow 3ax^2 + 6bx + 3c = 0 \quad at\ x = \alpha$

$\Rightarrow 3a\alpha^2 + 6b\alpha + 3c = 0$

$\Rightarrow a\alpha^2 + 2b\alpha + c = 0 \qquad \cdots(ii)$

$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \qquad \cdots(i)$

Multiplying (ii) by α

$\Rightarrow a\alpha^3 + 2b\alpha^2 + c\alpha = 0 \qquad \cdots(iii)$

Subtracting, we get

$b\alpha^2 + 2c\alpha + d = 0 \qquad \cdots(iv)$

$a\alpha^2 + 2b\alpha + c = 0 \qquad \cdots(ii) \quad \times b$

$b\alpha^2 + 2c\alpha + d = 0 \qquad \cdots(iv) \quad \times a$

Subtracting

$2(b^2 - ac)\alpha + bc - ad = 0$

$\Rightarrow \alpha = \dfrac{bc - ad}{2(ac - b^2)}$

<u>Location of Roots:</u>

Example (): The values of k for which the equation $x^3 - 3x - k = 0$ has only one real root

(a) $k = \pm 2$ (b) $-2 < k < 2$

(c) $k \in (-\infty, -2)$

(d) $k \in (2, \infty)$

Solution: options (c) & (d)

$x^3 - 3x - k = 0$

$\Rightarrow$ corresponding function is

$f(x) = x^3 - 3x - k.$

$f'(x) = 3x^2 - 3$

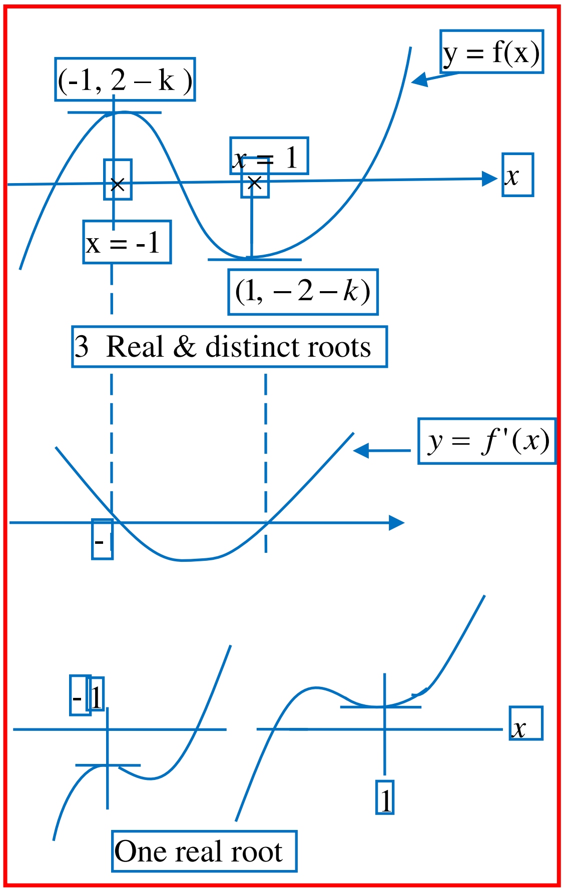
$f'(x) = 0 \quad \Rightarrow \quad x = \pm 1.$

$f''(x) = 6x$

$f''(1) = 6 \quad \text{(minima)}$

$f''(-1) = -6 \quad \text{(maxima)}$

$f(1) = -2 - k \quad \text{and} \quad f(-1) = 2 - k$



For one real root

either $f(-1) < 0$

$\Rightarrow \quad 2 - k < 0$

$\Rightarrow \quad k > 2$

or, $\quad f(1) > 0.$

$\Rightarrow \quad -2k > 0.$

$k < -2$

(9) Approximations and Errors:

Approximations:

Let $y = f(x)$ be the given function

Then from the definition of the derivatives,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) \approx \frac{f(x+h) - f(x)}{h} + \Delta \qquad \textit{where} \quad \Delta <<<<<$$

$$\Rightarrow hf'(x) \approx f(x+h) - f(x) + h\Delta$$

$h \to 0 \; and \; \Delta <<< \quad \Rightarrow \quad h\Delta <<<<<$ and is a very small quantity and can be neglected.

$$\Rightarrow hf'(x) \approx f(x+h) - f(x)$$
$$\Rightarrow f(x+h) \approx f(x) + hf'(x)$$

Approximation formula:

$$\boxed{f(x+h) \approx f(x) + h\,f'(x)}$$

Errors:

We have, $\dfrac{dy}{dx} = \lim_{\partial x \to 0} \dfrac{\partial y}{\partial x}$

Where ∂y is change in 'y' for corresponding change ∂x in 'x'

This is approximated as

$$\frac{dy}{dx} \approx \frac{\partial y}{\partial x} + \in \qquad\qquad \in <<<<$$

$\in <<<<$ can be neglected in comparison

Hence, $\dfrac{dy}{dx} \approx \dfrac{\partial y}{\partial x}$

$\Rightarrow$ $\partial y \approx \left(\dfrac{dy}{dx}\right)\partial x$

$\partial y, \partial x$ are defined as absolute errors in 'y' and 'x'.

$\dfrac{\partial y}{y}, \dfrac{\partial x}{x}$ are defined as relative errors

And $\left(\dfrac{\partial y}{y}\right) \times 100, \left(\dfrac{\partial x}{x}\right) \times 100$ are defined as percentage errors

(10) Application to Geometry

In this sub – concept, derivative is used

- As Rate Measurer
- For the determination of maximum and minimum values

(10) Application to Phusics

In this sub – concept, derivative is used

- As Rate Measurer

Examples: $\dfrac{dx}{dt} = Velocity\ (Rate\ of\ change\ of\ displacement$

$\dfrac{dv}{dt} = acceleration\ (Rate\ of\ change\ of\ velocity$